Let (a_n) be a bounded sequence of real numbers. Let us remind the notations

$$R_n := \{a_k \, : \, k \geq n\}, \quad s_n := \sup R_n, \quad m_n := \inf R_n.$$

Let us also recall the following definition:

Definition. Limit superior of (a_n) is defined as

$$\limsup_{n\to\infty} a_n := \lim_{n\to\infty} s_n = \inf s_n.$$

Limit inferior of (a_n) is defined as

$$\liminf_{n\to\infty} a_n := \lim_{n\to\infty} m_n = \sup m_n.$$

Theorem. *A sequence of real numbers converges if and only if it is a Cauchy sequence.*

Proof. $(\Rightarrow)$ Let $a = \lim_{n\to\infty} a_n$. Fix $\varepsilon > 0$. Then $\exists N$ such that $n \geq N \implies |a_n - a| < \varepsilon/2$. Take $m, \ n \geq N$. Then

$$|a_m - a_n| \geq |a_m - a| + |a_n - a| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

$(\Leftarrow)$ Assume that (a_n) is a Cauchy sequence and fix $\varepsilon > 0$. Then one can find N so that $m, \ n \geq N \implies |a_m - a_n| < \varepsilon/2$. In particular, if $m \geq n$, $a_N - \varepsilon/2 < a_m < a_N + \varepsilon/2$.

Thus $a_N + \varepsilon/2$ is an upper bound for R_N, and $a_N - \varepsilon/2$ is a lower bound for R_N. Thus

$$a_N - \varepsilon/2 \leq \inf R_N = m_N \leq s_N = \sup R_N \leq a_N + \varepsilon/2.$$

Thus

$$\limsup_{n\to\infty} a_n - \liminf_{n\to\infty} a_n \leq a_N + \varepsilon/2 - (a_N - \varepsilon/2) = \varepsilon.$$

Since this is true for any $\varepsilon > 0$,

$$\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n.$$

Thus (a_n) converges. $\square$